0.6 TRANSFORMATIONS OF FUNCTIONS

You are now familiar with a long list of functions: polynomials, rational functions, trigonometric functions, exponentials and logarithms. One important goal of this course is to more fully understand the properties of these functions. To a large extent, you will build your understanding by examining a few key properties of functions.

We expand on our list of functions by combining them. We begin in a straight-forward fashion with Definition 6.1.

DEFINITION 6.1

Suppose that $f(x)$ and $g(x)$ are functions with domains D_1 and D_2, respectively. The functions $f+g$, $f-g$ and $f \cdot g$ are defined by

$$(f+g)(x) = f(x) + g(x),$$

$$(f-g)(x) = f(x) - g(x)$$

and

$$(f \cdot g)(x) = f(x) \cdot g(x),$$

for all x in $D_1 \cap D_2$ (i.e., $x \in D_1$, *and* $x \in D_2$). The function $\dfrac{f}{g}$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

for all x in $D_1 \cap D_2$ such that $g(x) \neq 0$.

In example 6.1, we examine various combinations of several simple functions.

EXAMPLE 6.1 Combinations of Functions

If $f(x) = x - 3$ and $g(x) = \sqrt{x-1}$, determine the functions $f+g$, $3f-g$ and $\dfrac{f}{g}$, stating the domains of each.

Solution First, note that the domain of f is the entire real line and the domain of g is the set of all $x \geq 1$. Now,

$$(f+g)(x) = x - 3 + \sqrt{x-1}$$

and

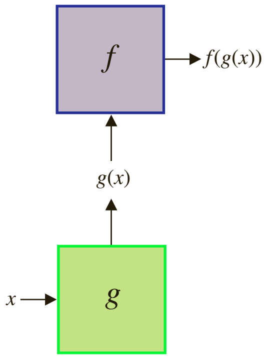
$$(3f-g)(x) = 3(x-3) - \sqrt{x-1} = 3x - 9 - \sqrt{x-1}.$$

Notice that the domain of both $(f+g)$ and $(3f-g)$ is $\{x \mid x \geq 1\}$. For

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-3}{\sqrt{x-1}},$$

the domain is $\{x \mid x > 1\}$, where we have added the restriction $x \neq 1$ to avoid dividing by 0. ■

Definition 6.1 and example 6.1 show us how to do arithmetic with functions. An operation on functions that does not directly correspond to arithmetic is the *composition* of two functions.



$(f \circ g)(x) = f(g(x))$

DEFINITION 6.2

The **composition** of functions f and g, written $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x)),$$

for all x such that x is in the domain of g and $g(x)$ is in the domain of f.

The composition of two functions is a two-step process, as indicated in the margin schematic. Be careful to notice what this definition is saying. In particular, for $f(g(x))$ to be defined, you first need $g(x)$ to be defined, so x must be in the domain of g. Next, f must be defined at the point $g(x)$, so that the number $g(x)$ will need to be in the domain of f.

EXAMPLE 6.2 Finding the Composition of Two Functions

For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$, find the compositions $f \circ g$ and $g \circ f$ and identify the domain of each.

Solution First, we have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(\sqrt{x-2}) \\ &= (\sqrt{x-2})^2 + 1 = x - 2 + 1 = x - 1.\end{aligned}$$

It's tempting to write that the domain of $f \circ g$ is the entire real line, but look more carefully. Note that for x to be in the domain of g, we must have $x \geq 2$. The domain of f is the whole real line, so this places no further restrictions on the domain of $f \circ g$. Even though the final expression $x - 1$ is defined for all x, the domain of $(f \circ g)$ is $\{x | x \geq 2\}$.

For the second composition,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(x^2 + 1) \\ &= \sqrt{(x^2+1) - 2} = \sqrt{x^2 - 1}.\end{aligned}$$

The resulting square root requires $x^2 - 1 \geq 0$ or $|x| \geq 1$. Since the "inside" function f is defined for all x, the domain of $g \circ f$ is $\{x \in \mathbb{R} \,\big|\, |x| \geq 1\}$, which we write in interval notation as $(-\infty, -1] \cup [1, \infty)$. ■

As you progress through the calculus, you will often find yourself needing to recognize that a given function is a composition of simpler functions. For now, it is an important skill to practice.

EXAMPLE 6.3 Identifying Compositions of Functions

Identify functions f and g such that the given function can be written as $(f \circ g)(x)$ for each of (a) $\sqrt{x^2+1}$, (b) $(\sqrt{x}+1)^2$, (c) $\sin x^2$ and (d) $\cos^2 x$. Note that more than one answer is possible for each function.

Solution (a) Notice that $x^2 + 1$ is *inside* the square root. So, one choice is to have $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$.

(b) Here, $\sqrt{x}+1$ is *inside* the square. So, one choice is $g(x)=\sqrt{x}+1$ and $f(x)=x^2$.

(c) The function can be rewritten as $\sin(x^2)$, with x^2 clearly *inside* the sine function. Then, $g(x)=x^2$ and $f(x)=\sin x$ is one choice.

(d) The function as written is shorthand for $(\cos x)^2$. So, one choice is $g(x)=\cos x$ and $f(x)=x^2$. ■

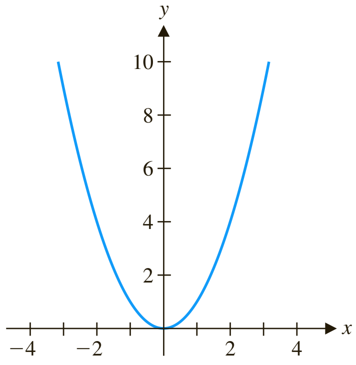


FIGURE 0.76a
$y=x^2$

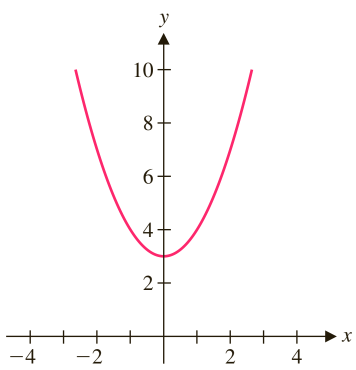


FIGURE 0.76b
$y=x^2+3$

In general, it is quite difficult to take the graphs of $f(x)$ and $g(x)$ and produce the graph of $f(g(x))$. If one of the functions f and g is linear, however, there is a simple graphical procedure for graphing the composition. Such **linear transformations** are explored in the remainder of this section.

The first case is to take the graph of $f(x)$ and produce the graph of $f(x)+c$ for some constant c. You should be able to deduce the general result from example 6.4.

EXAMPLE 6.4 Vertical Translation of a Graph

Graph $y=x^2$ and $y=x^2+3$; compare and contrast the graphs.

Solution You can probably sketch these by hand. You should get graphs like those in Figures 0.76a and 0.76b. Both figures show parabolas opening upward. The main obvious difference is that x^2 has a y-intercept of 0 and x^2+3 has a y-intercept of 3. In fact, for *any* given value of x, the point on the graph of $y=x^2+3$ will be plotted exactly 3 units higher than the corresponding point on the graph of $y=x^2$. This is shown in Figure 0.77a.

In Figure 0.77b, the two graphs are shown on the same set of axes. To many people, it does not look like the top graph is the same as the bottom graph moved up 3 units.

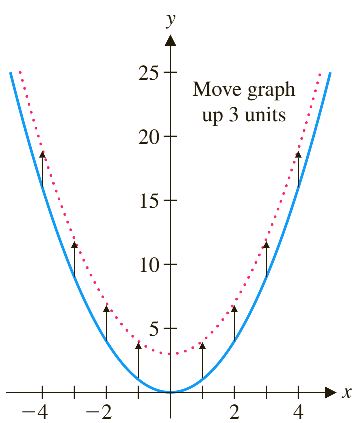


FIGURE 0.77a
Translate graph up

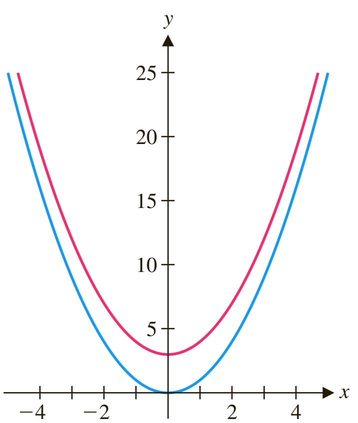


FIGURE 0.77b
$y=x^2$ and $y=x^2+3$

This is an unfortunate optical illusion. Humans usually mentally judge distance between curves as the shortest distance between the curves. For these parabolas, the shortest distance is vertical at $x=0$ but becomes increasingly horizontal as you move away from the y-axis. The distance of 3 between the parabolas is measured *vertically*. ■

In general, the graph of $y=f(x)+c$ is the same as the graph of $f(x)$ shifted up (if $c>0$) or down (if $c<0$) by $|c|$ units. We usually refer to $f(x)+c$ as a **vertical translation** (up or down, by $|c|$ units).

In example 6.5, we explore what happens if a constant is added to x.

EXAMPLE 6.5 A Horizontal Translation

Compare and contrast the graphs of $y = x^2$ and $y = (x-1)^2$.

Solution The graphs are shown in Figures 0.78a and 0.78b, respectively.

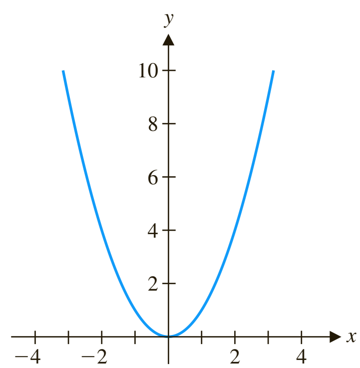


FIGURE 0.78a
$y = x^2$

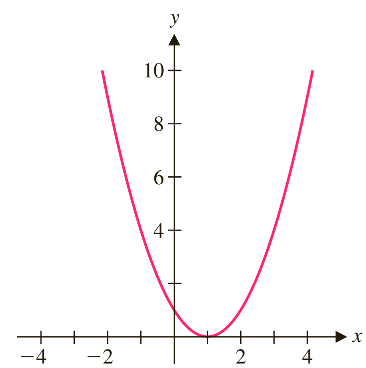


FIGURE 0.78b
$y = (x-1)^2$

Notice that the graph of $y = (x-1)^2$ appears to be the same as the graph of $y = x^2$, except that it is shifted 1 unit to the right. This should make sense for the following reason. Pick a value of x, say, $x = 13$. The value of $(x-1)^2$ at $x = 13$ is 12^2, the same as the value of x^2 at $x = 12$, 1 unit to the left. Observe that this same pattern holds for any x you choose. A simultaneous plot of the two functions (see Figure 0.79) shows this. ■

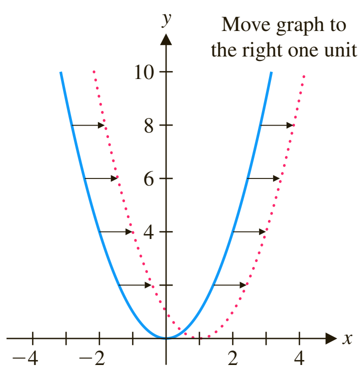


FIGURE 0.79
Translation to the right

In general, for $c > 0$, the graph of $y = f(x-c)$ is the same as the graph of $y = f(x)$ shifted c units to the right. Likewise (again, for $c > 0$), you get the graph of $f(x+c)$ by moving the graph of $y = f(x)$ to the left c units. We usually refer to $f(x-c)$ and $f(x+c)$ as **horizontal translations** (to the right and left, respectively, by c units).

To avoid confusion on which way to translate the graph of $y = f(x)$, focus on what makes the argument (the quantity inside the parentheses) zero. For $f(x)$, this is $x = 0$, but for $f(x-c)$ you must have $x = c$ to get $f(0)$ [i.e., the same y-value as $f(x)$ when $x = 0$]. This says that the point on the graph of $y = f(x)$ at $x = 0$ corresponds to the point on the graph of $y = f(x-c)$ at $x = c$.

EXAMPLE 6.6 Comparing Vertical and Horizontal Translations

Given the graph of $y = f(x)$ shown in Figure 0.80a, sketch the graphs of $y = f(x) - 2$ and $y = f(x-2)$.

Solution To graph $y = f(x) - 2$, simply translate the original graph down 2 units, as shown in Figure 0.80b. To graph $y = f(x-2)$, simply translate the original graph to

the right 2 units (so that the x-intercept at $x = 0$ in the original graph corresponds to an x-intercept at $x = 2$ in the translated graph), as seen in Figure 0.80c.

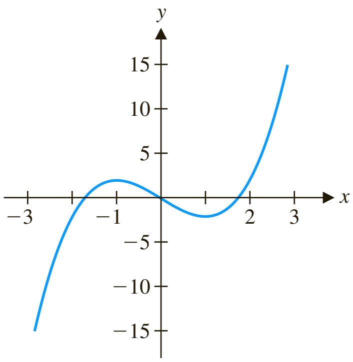


FIGURE 0.80a
$y = f(x)$

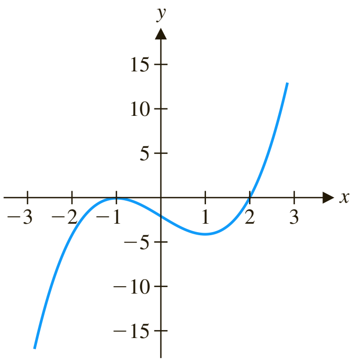


FIGURE 0.80b
$y = f(x) - 2$

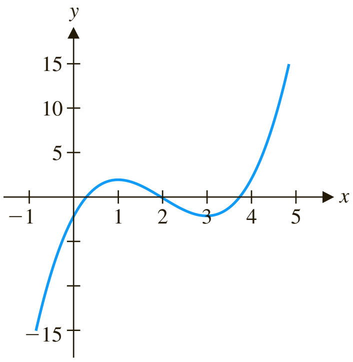


FIGURE 0.80c
$y = f(x - 2)$

Example 6.7 explores the effect of multiplying or dividing x or y by a constant.

EXAMPLE 6.7 Comparing Some Related Graphs

Compare and contrast the graphs of $y = x^2 - 1$, $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$.

Solution The first two graphs are shown in Figures 0.81a and 0.81b, respectively.

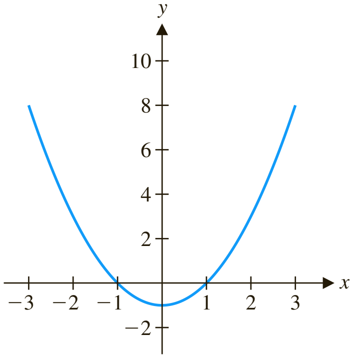


FIGURE 0.81a
$y = x^2 - 1$

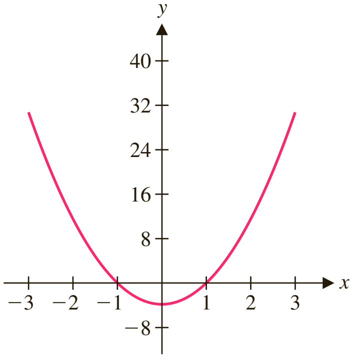


FIGURE 0.81b
$y = 4(x^2 - 1)$

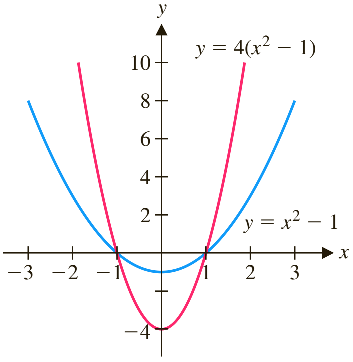


FIGURE 0.81c
$y = x^2 - 1$ and $y = 4(x^2 - 1)$

These graphs look identical until you compare the scales on the y-axes. The scale in Figure 0.81b is four times as large, reflecting the multiplication of the original function by 4. The effect looks different when the functions are plotted on the same scale, as in Figure 0.81c. Here, the parabola $y = 4(x^2 - 1)$ looks thinner and has a different y-intercept. Note that the x-intercepts remain the same. (Why would that be?)

The graphs of $y = x^2 - 1$ and $y = (4x)^2 - 1$ are shown in Figures 0.82a and 0.82b, respectively.

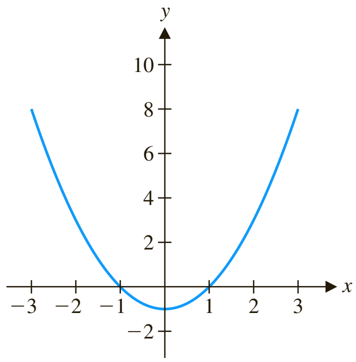


FIGURE 0.82a
$y = x^2 - 1$

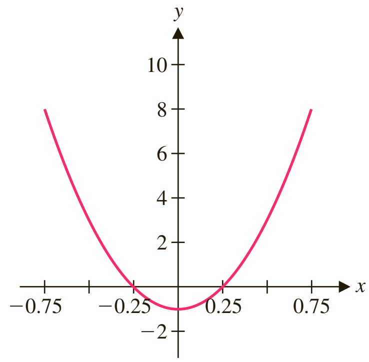


FIGURE 0.82b
$y = (4x)^2 - 1$

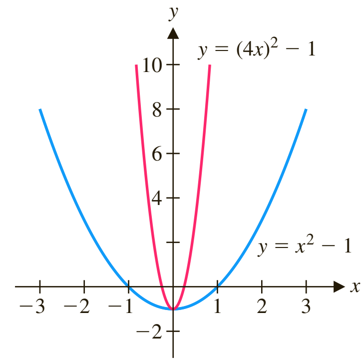


FIGURE 0.82c
$y = x^2 - 1$ and $y = (4x)^2 - 1$

Can you spot the difference here? In this case, the x-scale has now changed, by the same factor of 4 as in the function. To see this, note that substituting $x = 1/4$ into $(4x)^2 - 1$ produces $(1)^2 - 1$, exactly the same as substituting $x = 1$ into the original function. When plotted on the same set of axes (as in Figure 0.82c), the parabola $y = (4x)^2 - 1$ looks thinner. Here, the x-intercepts are different, but the y-intercepts are the same. ■

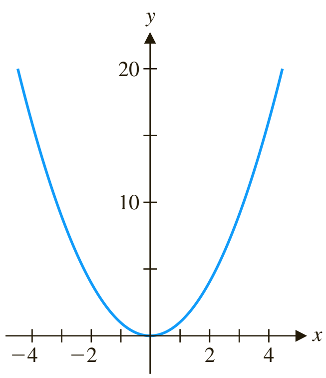


FIGURE 0.83a
$y = x^2$

We can generalize the observations made in example 6.7. Before reading our explanation, try to state a general rule for yourself. How are the graphs of the functions $cf(x)$ and $f(cx)$ related to the graph of $y = f(x)$?

Based on example 6.7, notice that to obtain a graph of $y = cf(x)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the y-axis by c. To obtain a graph of $y = f(cx)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the x-axis by $1/c$.

These basic rules can be combined to understand more complicated graphs.

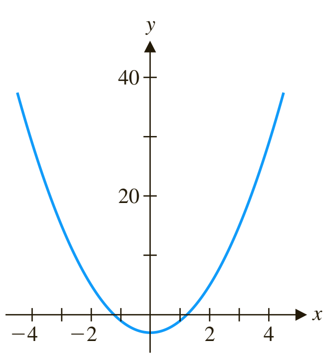


FIGURE 0.83b
$y = 2x^2 - 3$

EXAMPLE 6.8 A Translation and a Stretching

Describe how to get the graph of $y = 2x^2 - 3$ from the graph of $y = x^2$.

Solution You can get from x^2 to $2x^2 - 3$ by multiplying by 2 and then subtracting 3. In terms of the graph, this has the effect of multiplying the y-scale by 2 and then shifting the graph down by 3 units (see the graphs in Figures 0.83a and 0.83b). ■

EXAMPLE 6.9 A Translation in Both x- and y-Directions

Describe how to get the graph of $y = x^2 + 4x + 3$ from the graph of $y = x^2$.

Solution We can again relate this (and the graph of *every* quadratic) to the graph of $y = x^2$. We must first **complete the square.** Recall that in this process, you take the coefficient of x (4), divide by 2 ($4/2 = 2$) and square the result ($2^2 = 4$). Add and subtract this number and then, rewrite the x-terms as a perfect square. We have

$$y = x^2 + 4x + 3 = (x^2 + 4x + 4) - 4 + 3 = (x + 2)^2 - 1.$$

To graph this function, take the parabola $y = x^2$ (see Figure 0.84a) and translate the graph 2 units to the left and 1 unit down (see Figure 0.84b).

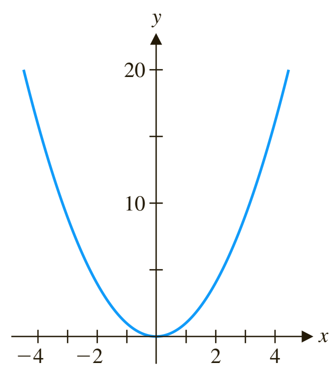


FIGURE 0.84a
$y = x^2$

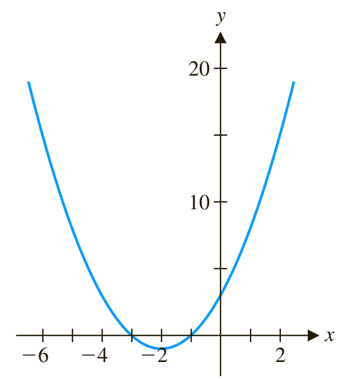


FIGURE 0.84b
$y = (x + 2)^2 - 1$

The following table summarizes our discoveries in this section.

Transformations of $f(x)$

Transformation	*Form*	*Effect on Graph*
Vertical translation	$f(x) + c$	$\|c\|$ units up ($c > 0$) or down ($c < 0$)
Horizontal translation	$f(x + c)$	$\|c\|$ units left ($c > 0$) or right ($c < 0$)
Vertical scale	$cf(x)\,(c > 0)$	multiply vertical scale by c
Horizontal scale	$f(cx)\,(c > 0)$	divide horizontal scale by c

You will explore additional transformations in the exercises.

EXERCISES 0.6

WRITING EXERCISES

1. The restricted domain of example 6.2 may be puzzling. Consider the following analogy. Suppose you have an airplane flight from New York to Los Angeles with a stop for refueling in Minneapolis. If bad weather has closed the airport in Minneapolis, explain why your flight will be canceled (or at least rerouted) even if the weather is great in New York and Los Angeles.

2. Explain why the graphs of $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$ in Figures 0.81c and 0.82c appear "thinner" than the graph of $y = x^2 - 1$.

3. As illustrated in example 6.9, completing the square can be used to rewrite any quadratic function in the form $a(x - d)^2 + e$. Using the transformation rules in this section, explain why this means that all parabolas (with $a > 0$) will look essentially the same.

4. Explain why the graph of $y = f(x + 4)$ is obtained by moving the graph of $y = f(x)$ four units to the left, instead of to the right.

In exercises 1–6, find the compositions $f \circ g$ and $g \circ f$, and identify their respective domains.

1. $f(x) = x + 1, \quad g(x) = \sqrt{x - 3}$
2. $f(x) = x - 2, \quad g(x) = \sqrt{x + 1}$
3. $f(x) = e^x, \quad g(x) = \ln x$
4. $f(x) = \sqrt{1 - x}, \quad g(x) = \ln x$
5. $f(x) = x^2 + 1, \quad g(x) = \sin x$
6. $f(x) = \dfrac{1}{x^2 - 1}, \quad g(x) = x^2 - 2$

In exercises 7–16, identify functions $f(x)$ and $g(x)$ such that the given function equals $(f \circ g)(x)$.

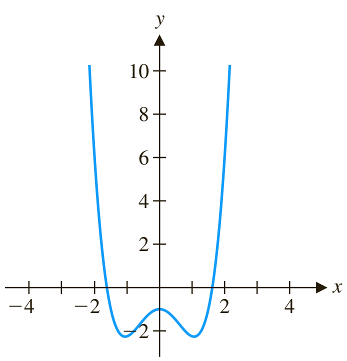
7. $\sqrt{x^4 + 1}$
8. $\sqrt[3]{x + 3}$
9. $\dfrac{1}{x^2 + 1}$
10. $\dfrac{1}{x^2} + 1$
11. $(4x + 1)^2 + 3$
12. $4(x + 1)^2 + 3$
13. $\sin^3 x$
14. $\sin x^3$
15. e^{x^2+1}
16. e^{4x-2}

In exercises 17–22, identify functions $f(x)$, $g(x)$ and $h(x)$ such that the given function equals $[f \circ (g \circ h)](x)$.

17. $\dfrac{3}{\sqrt{\sin x + 2}}$
18. $\sqrt{e^{4x} + 1}$
19. $\cos^3(4x - 2)$
20. $\ln \sqrt{x^2 + 1}$
21. $4e^{x^2} - 5$
22. $\left[\tan^{-1}(3x + 1)\right]^2$

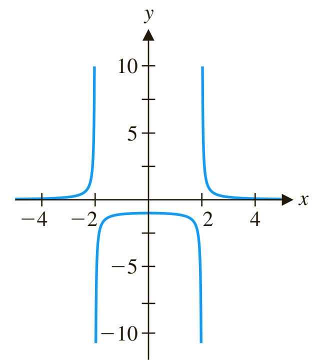
In exercises 23–30, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

23. $f(x) - 3$
24. $f(x + 2)$
25. $f(x - 3)$
26. $f(x) + 2$
27. $f(2x)$
28. $3f(x)$
29. $4f(x) - 1$
30. $3f(x + 2)$



In exercises 31–38, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

31. $f(x - 4)$
32. $f(x + 3)$
33. $f(2x)$
34. $f(2x - 4)$
35. $f(3x + 3)$
36. $3f(x)$
37. $2f(x) - 4$
38. $3f(x) + 3$



In exercises 39–44, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

39. $f(x) = x^2 + 2x + 1$
40. $f(x) = x^2 - 4x + 4$
41. $f(x) = x^2 + 2x + 4$
42. $f(x) = x^2 - 4x + 2$
43. $f(x) = 2x^2 + 4x + 4$
44. $f(x) = 3x^2 - 6x + 2$

In exercises 45–48, graph the given function and compare to the graph of $y = x^2 - 1$.

45. $f(x) = -2(x^2 - 1)$
46. $f(x) = -3(x^2 - 1)$
47. $f(x) = -3(x^2 - 1) + 2$
48. $f(x) = -2(x^2 - 1) - 1$

In exercises 49–52, graph the given function and compare to the graph of $y = (x - 1)^2 - 1 = x^2 - 2x$.

49. $f(x) = (-x)^2 - 2(-x)$
50. $f(x) = (-2x)^2 - 2(-2x)$
51. $f(x) = (-x)^2 - 2(-x) + 1$
52. $f(x) = (-3x)^2 - 2(-3x) - 3$

53. Based on exercises 45–48, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = cf(x)$ for $c < 0$.

54. Based on exercises 49–52, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = f(cx)$ for $c < 0$.

55. Sketch the graph of $y = |x|^3$. Explain why the graph of $y = |x|^3$ is identical to that of $y = x^3$ to the right of the y-axis. For $y = |x|^3$, describe how the graph to the left of the y-axis compares to the graph to the right of the y-axis. In general, describe how to draw the graph of $y = f(|x|)$ given the graph of $y = f(x)$.

56. For $y = x^3$, describe how the graph to the left of the y-axis compares to the graph to the right of the y-axis. Show that for $f(x) = x^3$, we have $f(-x) = -f(x)$. In general, if you have the graph of $y = f(x)$ to the right of the y-axis and $f(-x) = -f(x)$ for all x, describe how to graph $y = f(x)$ to the left of the y-axis.

57. **Iterations** of functions are important in a variety of applications. To iterate $f(x)$, start with an initial value x_0 and compute $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$ and so on. For example, with $f(x) = \cos x$ and $x_0 = 1$, the **iterates** are $x_1 = \cos 1 \approx 0.54$, $x_2 = \cos x_1 \approx \cos 0.54 \approx 0.86$, $x_3 \approx \cos 0.86 \approx 0.65$ and so on. Keep computing iterates and show that they get closer and closer to 0.739085. Then pick your own x_0 (any number you like) and show that the iterates with this new x_0 also converge to 0.739085.

58. Referring to exercise 57, show that the iterates of a function can be written as $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, $x_3 = f(f(f(x_0)))$ and so on. Graph $y = \cos(\cos x)$, $y = \cos(\cos(\cos x))$ and $y = \cos(\cos(\cos(\cos x)))$. The graphs should look more and more like a horizontal line. Use the result of exercise 57 to identify the limiting line.

59. Compute several iterates of $f(x) = \sin x$ (see exercise 57) with a variety of starting values. What happens to the iterates in the long run?

60. Repeat exercise 59 for $f(x) = x^2$.

61. In cases where the iterates of a function (see exercise 57) repeat a single number, that number is called a **fixed point.** Explain why any fixed point must be a solution of the equation $f(x) = x$. Find all fixed points of $f(x) = \cos x$ by solving the equation $\cos x = x$. Compare your results to that of exercise 57.

62. Find all fixed points of $f(x) = \sin x$ (see exercise 61). Compare your results to those of exercise 59.

EXPLORATORY EXERCISES

1. You have explored how completing the square can transform any quadratic function into the form $y = a(x - d)^2 + e$. We concluded that all parabolas with $a > 0$ look alike. To see that the same statement is not true of cubic polynomials, graph $y = x^3$ and $y = x^3 - 3x$. In this exercise, you will use completing the cube to determine how many different cubic graphs there are. To see what "completing the cube" would look like, first show that $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$. Use this result to transform the graph of $y = x^3$ into the graphs of (a) $y = x^3 - 3x^2 + 3x - 1$ and (b) $y = x^3 - 3x^2 + 3x + 2$. Show that you can't get a simple transformation to $y = x^3 - 3x^2 + 4x - 2$. However, show that $y = x^3 - 3x^2 + 4x - 2$ can be obtained from $y = x^3 + x$ by basic transformations. Show that the following statement is true: any cubic ($y = ax^3 + bx^2 + cx + d$) can be obtained with basic transformations from $y = ax^3 + kx$ for some constant k.

2. In many applications, it is important to take a section of a graph (e.g., some data) and extend it for predictions or other analysis. For example, suppose you have an electronic signal equal to $f(x) = 2x$ for $0 \le x \le 2$. To predict the value of the signal at $x = -1$, you would want to know whether the signal was periodic. If the signal is periodic, explain why $f(-1) = 2$ would be a good prediction. In some applications, you would assume that the function is *even.* That is, $f(x) = f(-x)$ for all x. In this case, you want $f(x) = 2(-x) = -2x$ for $-2 \le x \le 0$. Graph the *even extension* $f(x) = \begin{cases} -2x & \text{if } -2 \le x \le 0 \\ 2x & \text{if } 0 \le x \le 2 \end{cases}$. Find the even extension for (a) $f(x) = x^2 + 2x + 1, 0 \le x \le 2$ and (b) $f(x) = e^{-x}, 0 \le x \le 2$.

3. Similar to the even extension discussed in exploratory exercise 2, applications sometimes require a function to be *odd;* that is, $f(-x) = -f(x)$. For $f(x) = x^2$, $0 \le x \le 2$, the odd extension requires that for $-2 \le x \le 0$, $f(x) = -f(-x) = -(-x)^2 = -x^2$ so that $f(x) = \begin{cases} -x^2 & \text{if } -2 \le x \le 0 \\ x^2 & \text{if } 0 \le x \le 2 \end{cases}$. Graph $y = f(x)$ and discuss how to graphically rotate the right half of the graph to get the left half of the graph. Find the odd extension for (a) $f(x) = x^2 + 2x$, $0 \le x \le 2$ and (b) $f(x) = e^{-x} - 1$, $0 \le x \le 2$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Slope of a line	Parallel lines	Perpendicular lines
Domain	Intercepts	Zeros of a function
Graphing window	Local maximum	Vertical asymptote
Inverse function	One-to-one function	Periodic function
Sine function	Cosine function	Arcsine function
e	Exponential function	Logarithm
Composition		